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Effect of Centerbody Scattering on Advanced Open-Rotor Noise

Michael J. Kingan,* Christopher Powles,* and Rod H. Self†
University of Southampton, Southampton, England SO17 1BJ, United Kingdom

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Formulas for calculating the effect of centerbody scattering on the sound radiated from an advanced open rotor are presented. The effects of blade sweep and distributed blade loading are considered. Mach number effects are also implicitly included in the model. The work extends a previously published method and applies it to a practical situation in which scattering by the centerbody has a significant effect on the radiated sound field.

Nomenclature

B	=	number of blades on the propeller
B_k	=	disturbance periodicity (number of disturbances per revolution)
b	=	hub/centerbody radius, m
c	=	chord length, m
c_0	=	speed of sound, m.s^{-1}
f	=	magnitude of $\mathbf{f}$, Pa
$\mathbf{f}$	=	load per unit area (vector) exerted by the propeller blade on the air, Pa
$\tilde{\mathbf{f}}$	=	Fourier transform in the time domain of $\mathbf{f}$, Pa · s
G	=	time-domain acoustic Green's function
$\tilde{G}$	=	frequency-domain acoustic Green's function
J, $H^{(2)}$	=	Bessel functions
k	=	index associated with the unsteady loading on the rotor blades [see Eq. (4)]
M_r	=	Mach number of the local flow relative to the blade
M_x	=	axial Mach number of the airflow (in the positive x direction)
m	=	index associated with the rotor from which acoustic radiation occurs [see Eq. (5)]
$\hat{\mathbf{n}}$	=	outward-facing unit normal to the upper blade surface
p	=	acoustic pressure, Pa
r	=	source radial coordinate, m
S	=	closed surface on which loading is assumed to act
s	=	sweep, m
t	=	receiver time, s
U_x	=	axial speed of the airflow (in the positive x direction), m.s^{-1}
X	=	chordwise coordinate, m
x, y, z	=	Cartesian coordinates, m
$\mathbf{x}$	=	source coordinate (vector), m
$\hat{\mathbf{x}}$	=	unit vector in the x direction
x_o	=	observer axial coordinate, m
$\mathbf{x}_o$	=	observer coordinate (vector), m
$\lvert\mathbf{x}_o\rvert$	=	observer radius, m
α	=	blade stagger angle, rad
$\bar{\alpha}$	=	axial wave number, m^{-1}
Δf	=	magnitude of $\Delta\mathbf{f}$, Pa
$\Delta\mathbf{f}$	=	difference in $\mathbf{f}$ between the top and bottom blade surfaces (vector), Pa
$\Delta\tilde{f}$	=	Fourier transform in the time domain of Δf, Pa · s
δ	=	Dirac's delta function
κ	=	$\equiv \omega/c_o$, rad.m^{-1}
ν	=	$kB_k - mB$
θ	=	polar emission angle, rad
Σ	=	blade planform surface
τ	=	source time, s
ϕ	=	source azimuthal coordinate, rad
$\hat{\boldsymbol{\phi}}$	=	unit vector in the ϕ direction
ϕ_o	=	observer azimuthal coordinate, rad
ψ	=	azimuthal coordinate relating to the sweep and chordwise location of the source, rad
Ω	=	propeller rotational speed, rad.s^{-1}
Ω_k	=	disturbance rotational speed, in the negative azimuthal direction, rad.s^{-1}
ω	=	frequency, rad.s^{-1}
ω_{mk}	=	frequency of the tone associated with the m and k indices

I. Introduction

TRADITIONAL formulas for predicting the tones produced by a propeller assume that the rotor operates in a free field and, thus, neglect scattering from the surrounding structures; in particular, from the centerbody or hub. Glegg [1] derived formulas for predicting the far-field sound pressure produced by a propeller with the effect of reflections from the hub included. The propeller was modeled acoustically as a rotating point force, and the centerbody was represented by a rigid cylinder of infinite length. Glegg did not include the effect of propeller motion in his model. It was shown that scattering from the centerbody had a small effect on the tones produced by steady loading forces on the propeller blades, but it was significant for a particular type of impulsive loading, similar to that produced by the interaction of the tip vortex from an upstream propeller with the downstream propeller of a counter-rotatingl propeller (also known as an advanced open rotor or unducted fan).

This paper represents an extension of Glegg's work and presents formulas for predicting the tones produced by a propeller that, like Glegg's formulation, account for the effect of centerbody scattering by approximating the hub as an infinite rigid cylinder. However, the formulas extend the method to include most of the significant parameters and effects that would be required to make a full propeller noise prediction, including 1) an arbitrary radial and chordwise loading distribution, 2) the effect of blade sweep, and 3) the effect of constant axial flow on the radiated sound field. The formulas are then used to predict the tones produced by the interaction of a wake from the upstream propeller with the downstream propeller of a counter-rotating open rotor.

The motivation for this study was the renewed interest in advanced open-rotor engines among aircraft engine manufacturers. Advanced open rotors offer potentially significant reductions in fuel burn relative to current and future turbofan technologies, making them an

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*Research Fellow, Institute of Sound and Vibration Research, University Road.

†Lecturer, Institute of Sound and Vibration Research, University Road.

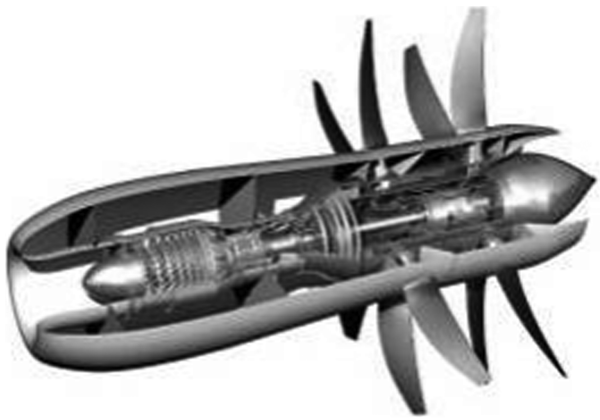

Fig. 1 Advanced open-rotor concept (courtesy of Rolls–Royce).

attractive alternative aircraft propulsor because of their relatively low fuel costs and emissions. A cutaway view showing an open-rotor concept is shown in Fig. 1.

II. Formulation

Goldstein [2] (Eq. 4.13) gives an expression for the sound pressure produced by an arbitrary load distribution $\mathbf{f}(\mathbf{x},\ \tau)$, exerted on the air, at point $\mathbf{x}$ and time τ. The load distribution is situated in a uniform isentropic flow of Mach number M_x aligned with the positive x direction and acts on a closed surface S that, in this case, corresponds to the propeller blade surface. Contributions from quadrupole and volume displacement sources are neglected. The expression uses a time-domain Green's function $G(\mathbf{x}_o,\ t|\mathbf{x},\ \tau)$ that satisfies the convected wave equation and a rigid boundary condition on the hub surface:

$$p(\mathbf{x}_o,\ t) = \int_{-\infty}^{\infty} \int_S \mathbf{f}(\mathbf{x},\ \tau)\cdot\nabla G(\mathbf{x}_o,\ t|\mathbf{x},\ \tau)\,\mathrm{d}S(\mathbf{x})\,\mathrm{d}\tau \tag{1}$$

Assuming that the propeller blades can be approximated as infinitely thin, the pressure on the upper and lower surfaces of the blades is assumed to act on, and be normal to, the blade chord line. This gives the net pressure exerted by the blades on the air as the pressure difference between the upper and lower blade surfaces, $\Delta\mathbf{f}(\mathbf{x},\ \tau) = \hat{\mathbf{n}}[f(\mathbf{x}^+,\ \tau) - f(\mathbf{x}^-,\ \tau)]$, where f is the magnitude of the load on either the upper $\mathbf{x}^+$ or lower, $\mathbf{x}^-$ blade surface and $\hat{\mathbf{n}}$ is the outward-facing unit normal to the upper blade surface. Using this simplification, the integration over both the upper and lower blade surfaces may be replaced by the integration over the blade planform Σ. Taking the Fourier transform of Eq. (1) with respect to time and making use of the convolution theorem (noting that $G(\mathbf{x}_o,\ t|\mathbf{x},\ \tau)$ is a function of $t-\tau$) yields,

$$\tilde{p}(\mathbf{x}_o) = \int_\Sigma \Delta\tilde{\mathbf{f}}(\mathbf{x})\cdot\nabla\tilde{G}(\mathbf{x}_o|\mathbf{x})\,\mathrm{d}\Sigma(\mathbf{x}) \tag{2}$$

where $\tilde{p}(\mathbf{x}_o)$, $\Delta\tilde{\mathbf{f}}(\mathbf{x})$, and$\nabla\tilde{G}(\mathbf{x}_o|\mathbf{x})$ are the Fourier transforms (with respect to time) of $p(\mathbf{x}_o,\ t)$, $\mathbf{f}(\mathbf{x},\ t)$, and $\nabla G(\mathbf{x}_o,\ t|\mathbf{x},\ 0)$. The coordinate system and loading forces exerted by the blade on the air are shown in Fig. 2.

In cylindrical coordinates $\{x,\ r,\ \phi\}$, each blade exerts a force on an element of air of span $\mathrm{d}r$ and chord $\mathrm{d}X$,

$$\Delta\mathbf{f}(\mathbf{x},\ \tau) = -\Delta f(\mathbf{x},\ \tau)[\hat{\mathbf{x}}\sin\alpha + \hat{\boldsymbol{\phi}}\cos\alpha]\,\mathrm{d}r\,\mathrm{d}X \tag{3}$$

where $\hat{\mathbf{x}}$ and $\hat{\boldsymbol{\phi}}$ are unit vectors in the axial and azimuthal directions, respectively, s is the local blade sweep, and α is the blade stagger angle, which is taken to be aligned with the local flow direction.

It is assumed that loading on the propeller blades is caused by a disturbance with azimuthal periodicity of $2\pi/B_k$ rad and rotates at an angular speed of $-\Omega_k$ rad/s. The loading can thus be expressed as a Fourier series:

$$\Delta f(\mathbf{x},\ \tau) = \sum_{k=-\infty}^{\infty} \Delta f_k \exp\{ikB_k(\phi + \Omega_k\tau)\} \tag{4}$$

As the propeller has B blades that rotate at angular speed Ω, the force exerted by the propeller on the air can be expressed as

$$\Delta f(\mathbf{x},\ \tau) = \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \Delta f_k \exp\{ikB_k(\phi + \Omega_k\tau)\}\delta\left(\phi - \Omega\tau + \frac{2\pi}{B}m\right) \tag{5}$$

Expressing Eq. (5) as a Fourier series yields:

$$\Delta f(\mathbf{x},\ \tau) = \frac{B}{2\pi}\sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \Delta f_k \exp\{i\omega_{mk}\tau + i\nu\phi\}; \qquad \nu = kB_k - mB; \qquad \omega_{mk} = mB\Omega + kB_k\Omega_k \tag{6}$$

Taking the Fourier transform of Eq. (6) with respect to τ gives:

$$\Delta\tilde{f}(\mathbf{x}) \equiv \int_{-\infty}^{\infty} \Delta f(\mathbf{x},\ \tau)\exp\{-i\omega\tau\}\,\mathrm{d}\tau = \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} B\,\Delta f_k \exp\{i\nu\phi\}\delta(\omega - \omega_{mk}) \tag{7}$$

The frequency-domain Green's function $\tilde{G}(\mathbf{x}_o|\mathbf{x})$ exterior to a rigid cylinder of infinite length and radius b, for a source at radius r and azimuthal angle $\phi - \psi$, is given by (this result is derived in Appendix B)

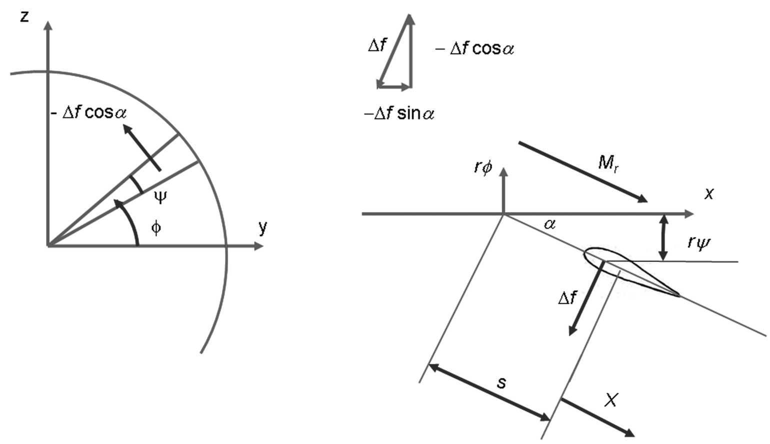

Fig. 2 Propeller geometry for a blade rotating in the positive ϕ direction: (left) the component forces and blade position in the nominal disc plane and (right) a single blade element.

$$\tilde{G}(\mathbf{x}_o|\mathbf{x}) = -\frac{i}{8\pi}\sum_{n=-\infty}^{\infty}\exp\{in(\phi-\psi-\phi_o)\}$$
$$\times\int_{-\infty}^{\infty}\exp\{i\bar{\alpha}(x-x_o)\}[J_n(\beta r_<)$$
$$-A_n(\bar{\alpha},\,\omega)H_n^{(2)}(\beta r_<)]H_n^{(2)}(\beta r_>)\,\mathrm{d}\bar{\alpha}$$

where

$$A_n(\bar{\alpha},\,\omega) = \frac{J_n'(\beta b)}{H_n^{(2)\prime}(\beta b)};\qquad \beta^2 = (\kappa-\bar{\alpha}M_x)^2-\bar{\alpha}^2$$

$$\kappa = \omega/c_0;\qquad M_x = U_x/c_0;\qquad r_> = \begin{Bmatrix} r_o & \text{if } r_o > r \\ r & \text{if } r_o < r \end{Bmatrix}$$

$$r_< = \begin{Bmatrix} r & \text{if } r_o > r \\ r_o & \text{if } r_o < r \end{Bmatrix}$$

For $|\mathbf{x}_o| \to \infty$, $r_o > r$ for all but grazing angles; thus,

$$\tilde{G}(\mathbf{x}_o|\mathbf{x}) = -\frac{i}{8\pi}\sum_{n=-\infty}^{\infty}\exp\{in(\phi-\psi-\phi_o)\}$$
$$\times\int_{-\infty}^{\infty}\exp\{i\bar{\alpha}(x-x_o)\}[J_n(\beta r)$$
$$-A_n(\bar{\alpha},\,\omega)H_n^{(2)}(\beta r)]H_n^{(2)}(\beta r_o)\,\mathrm{d}\bar{\alpha} \tag{8}$$

Evaluating the gradient of the Green's function, substituting the result and Eqs. (3) and (7) into Eq. (2), and noting that an element of the blade planform area is defined as $\mathrm{d}\Sigma = \mathrm{d}\phi\,\mathrm{d}X\,\mathrm{d}r$ and the blade chord length is denoted c gives

$$\tilde{p}(\mathbf{x}_o) = -\sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\int_b^{R_t}\int_{-c/2}^{c/2}\int_0^{2\pi}\frac{B}{8\pi}\Delta f_k\exp\{i(\nu+n)\phi$$
$$-in(\psi+\phi_o)\}I_{\bar{\alpha}}\,\mathrm{d}\phi\,\mathrm{d}X\,\mathrm{d}r\delta(\omega-\omega_{mk}) \tag{9}$$

where

$$I_{\bar{\alpha}} = \int_{-\infty}^{\infty}\left[\bar{\alpha}\sin\alpha+\frac{n}{r}\cos\alpha\right]\exp\{i\bar{\alpha}(x-x_o)\}\Theta H_n^{(2)}(\beta r_o)\,\mathrm{d}\bar{\alpha} \tag{10}$$

and

$$\Theta = [J_n(\beta r)-A_n(\bar{\alpha},\,\omega)H_n^{(2)}(\beta r)]$$

The integral over ϕ is zero for all values of n except $n = -\nu$; thus, the triple summation reduces to a double summation. Taking the inverse Fourier transform in time yields:

$$p(\mathbf{x}_o,\,t) = -\frac{B}{8\pi}\sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\int_b^{R_t}\int_{-c/2}^{c/2}\Delta f_k\exp\{i\omega_{mk}t$$
$$+i\nu(\psi+\phi_o)\}I_{\bar{\alpha}}\,\mathrm{d}X\,\mathrm{d}r \tag{11}$$

A good approximation for the integral defined by Eq. (10) can be found using the method of stationary phase (see Appendix A). Substituting the result into Eq. (11), using $\psi = \sin\alpha(s+X)/r$ and $x = (s+X)\cos\alpha$ (deduced from Fig. 2), and performing the integration over r and X yields,

$$p(\mathbf{x}_o,\,t) \sim \frac{B}{4\pi|\mathbf{x}_o|(1-M_x\cos\theta)}\sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\exp\left\{i\omega_{mk}\left(t-\frac{|\mathbf{x}_o|}{c_0}\right)\right.$$
$$\left.+i\nu\left(\phi_o+\frac{\pi}{2}\right)\right\}I \tag{12}$$

where I is an integral along the blade radius between the blade hub and tip, which is defined as

$$I = \int_b^{R_t}S(r)\exp\{-i\phi_s\}\bar{\Theta}\,\mathrm{d}r \tag{13}$$

where ϕ_s is a phase term associated with the local blade sweep s,

$$\phi_s = \left(\frac{\omega_{mk}/c_0}{(1-M_x\cos\theta)}\cos\theta\cos\alpha-\nu\frac{\sin\alpha}{r}\right)s \tag{14}$$

$S(r)$ is defined as

$$S(r) = \left[i\frac{\omega_{mk}/c_0}{(1-M_x\cos\theta)}\cos\theta\sin\alpha+i\cos\alpha\frac{\nu}{r}\right]\frac{\mathrm{d}L}{\mathrm{d}r} \tag{15}$$

$\mathrm{d}L/\mathrm{d}r$ is defined as

$$\frac{\mathrm{d}L}{\mathrm{d}r} = c\int_{-1/2}^{1/2}\Delta f_k\exp\{-ik_x\tilde{X}\}\,\mathrm{d}\tilde{X};\qquad \tilde{X} = \frac{X}{c} \tag{16}$$

k_x is a chordwise wave number,

$$k_x = \left(\frac{\omega_{mk}/c_0}{(1-M_x\cos\theta)}\cos\theta\cos\alpha-\nu\frac{\sin\alpha}{r}\right)c \tag{17}$$

and

$$\bar{\Theta} = J_\nu\left(\frac{\omega_{mk}/c_0}{1-M_x\cos\theta}r\sin\theta\right)$$
$$-\frac{J_\nu'\{[(\omega_{mk}/c_0)/(1-M_x\cos\theta)]b\sin\theta\}}{H_\nu^{(2)\prime}\{[(\omega_{mk}/c_0)/(1-M_x\cos\theta)]b\sin\theta\}}$$
$$\times H_\nu^{(2)}\left(\frac{\omega_{mk}/c_0}{1-M_x\cos\theta}r\sin\theta\right) \tag{18}$$

For tones produced by the interaction of a rotating disturbance with a counter-rotating propeller, terms in the summation in Eq. (12), for which the signs of k and m are different, can be ignored, as the mode phase speed will be subsonic across the whole span of the blade; therefore, the Bessel functions will be exponentially small ([3], p. 91). The $k = 0$ terms correspond to rotor-alone tones associated with the rotating steady loading on the rotor blades. Terms for which $k \neq 0$ correspond to interaction tones, which are due to the unsteady loading on the rotor blades produced (for example) by the interaction of the blades with a flow distortion (such as the rotating distortion produced by the upstream rotor wakes).

III. Loading due to Wake from Upstream Rotor onto Downstream Rotor

The sound produced by the viscous wake shed from the upstream propeller impinging on the downstream propeller of a counter-rotating open rotor is considered. The equations derived in Sec. II, for calculating the radiated sound pressure, require a knowledge of the unsteady loading distribution on the propeller blades. The method described in [3] (chapter 5) was used to calculate this unsteady loading. The method expresses the velocity deficit in the wake of the upstream propeller as a series of harmonic gusts, and the unsteady loading produced by each of these gusts is calculated using a well-known blade response function.

IV. Results

In this section, the models developed in this paper are used to assess the importance of scattering from the centerbody for tones produced by the interaction of the viscous wake shed from the upstream propeller, with the downstream propeller of a counter-rotating advanced open rotor.

The advanced open-rotor configuration being considered has 11 blades on the front rotor, 9 blades on the rear rotor, a front-rotor hub-to-tip ratio of 0.425, a front-rotor solidity of 0.454, and a front-rotor tip radius of 0.313 m. The ratio of the rotational speeds of the front rotor to the rear rotor was 0.99, and the rear rotor had a tip radius that was equal to 0.97 of the tip radius of the front rotor, and it had a solidity of 0.371. The operating condition was for an axial inflow Mach number of 0.2, and the helical tip Mach number of the first rotor was 0.754.

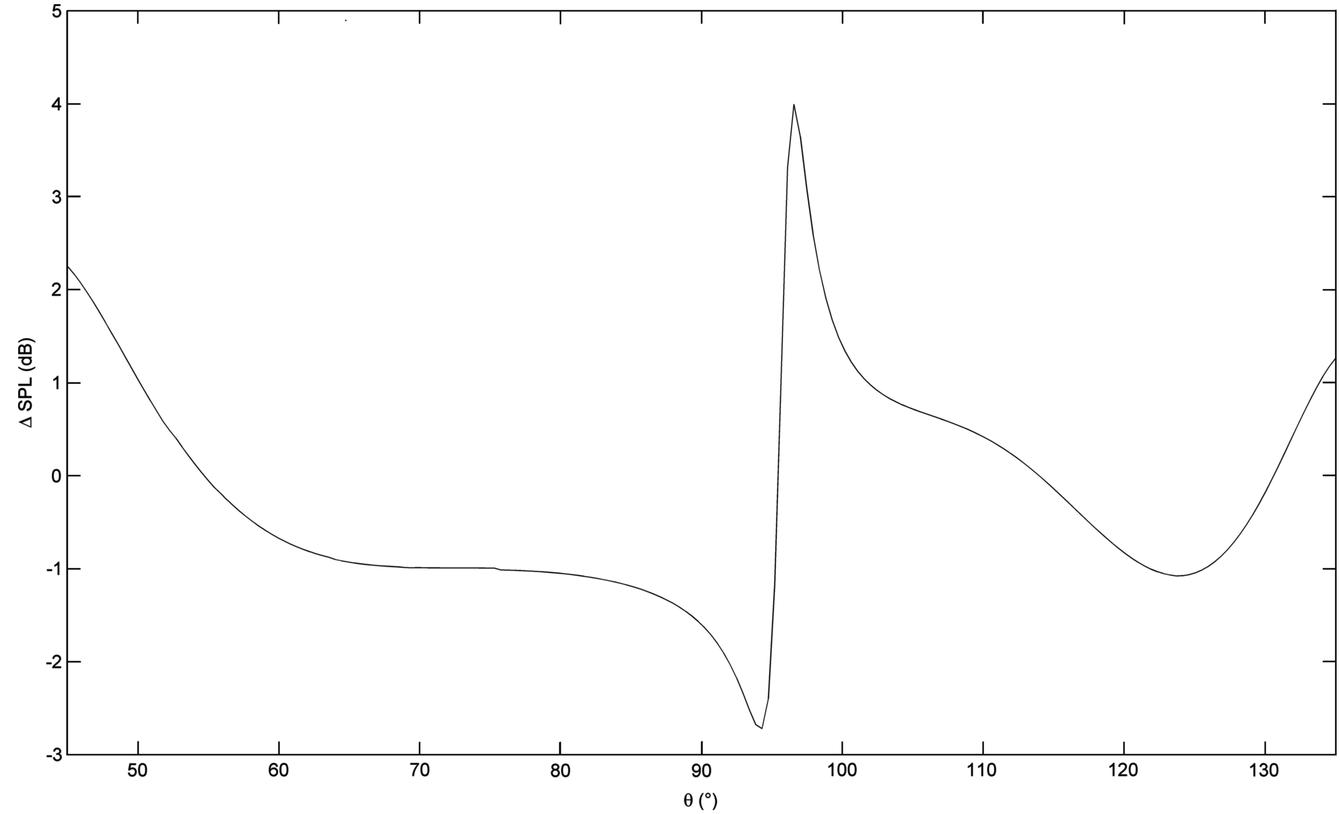

Fig. 3 Scattered SPL (dB) as a function of polar emission angle for the tone corresponding to $m = 1$ and $k = 1$.

A scattered sound pressure level (denoted as ΔSPL) is defined as the difference between the sound pressure level (SPL) calculated with hub scattering and the sound pressure level calculated by assuming no hub scattering. In Figs. 3 and 4, ΔSPL is plotted against polar emission angle θ for two tones. Figure 5 plots ΔSPL versus frequency for all rotor–rotor viscous wake interaction tones at polar emission angle $\theta = 90$ deg.

The results clearly show that the centerbody has a nonnegligible effect on the radiated sound field. Indeed, Fig. 5 shows that including the hub scattering in a noise prediction can have an effect of more

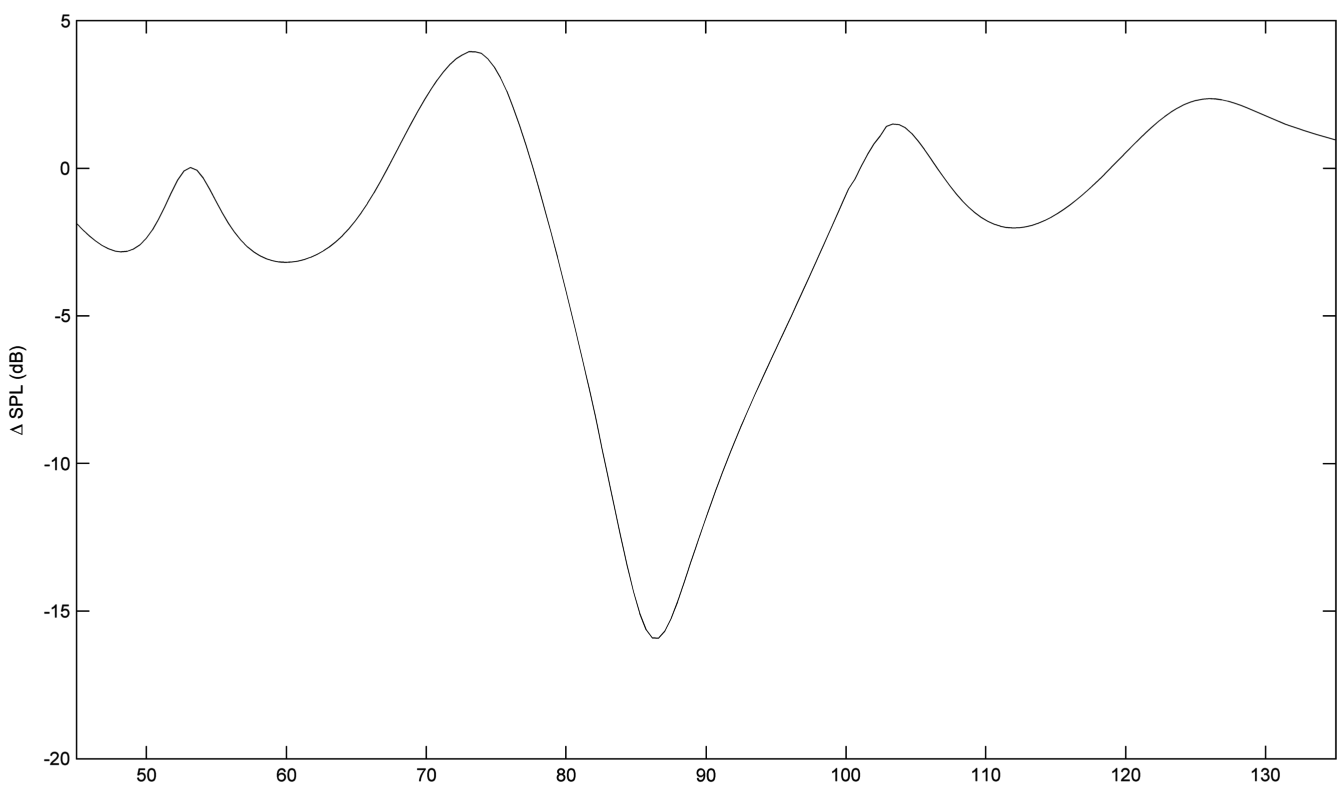

Fig. 4 Scattered SPL (dB) as a function of polar emission angle for the tone corresponding to $m = 4$ and $k = 2$.

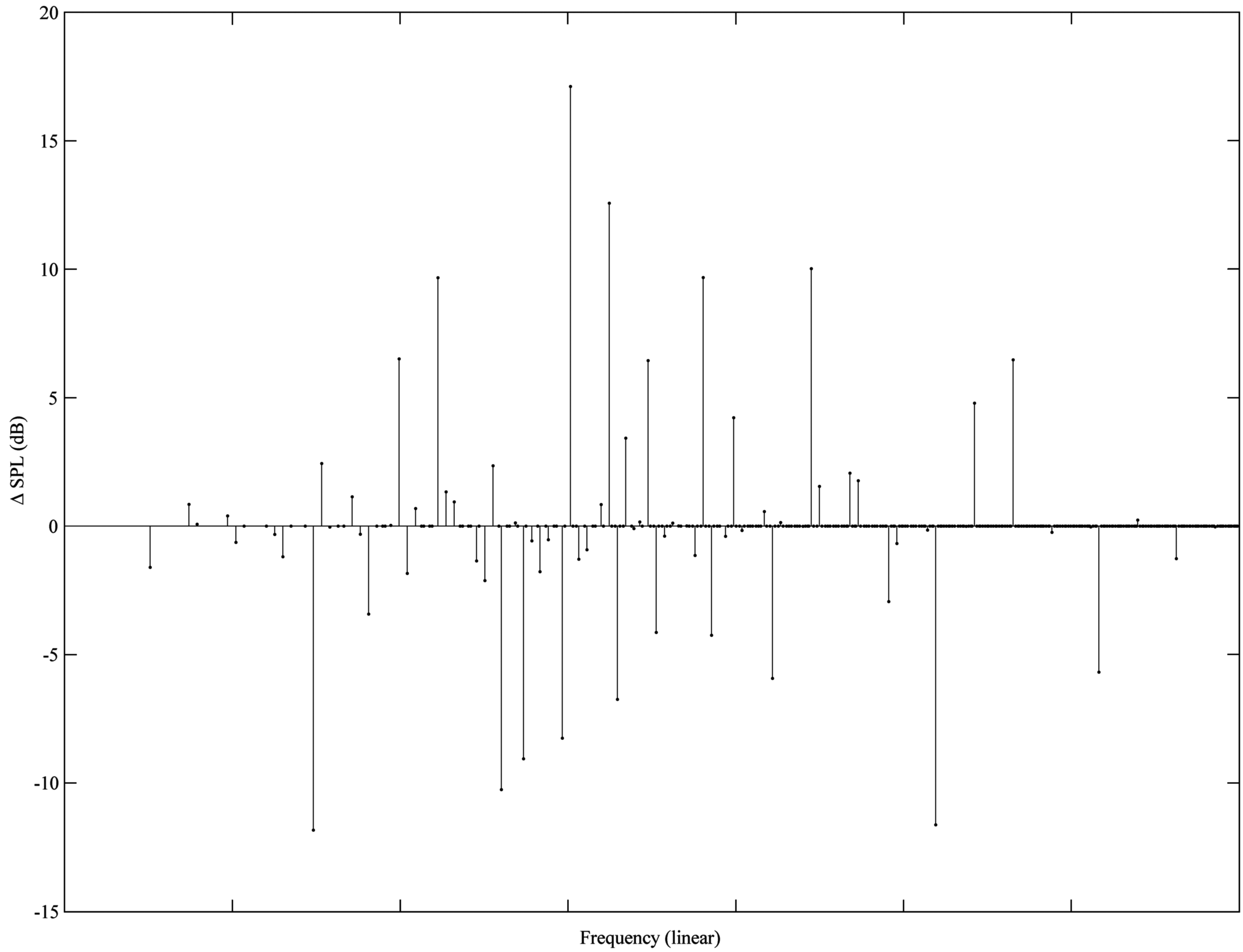

Fig. 5 Scattered SPL (dB) versus frequency for all rotor–rotor interaction tones at polar emission angle $\theta = 90$ deg.

than 15 dB, relative to a prediction that assumes free-field radiation. Although, it should be emphasized that a significant number of tones, particularly at high frequencies, remained unaffected by centerbody scattering for this particular case.

V. Conclusions

This paper has presented formulas for predicting the sound pressure level from a propeller or advanced open rotor, which includes the effect of scattering from the centerbody. The scattering effect was implicitly included in the formulas using a tailored Green's function, which satisfied the convected acoustic wave equation and had a zero normal derivative on the centerbody surface. The centerbody was modeled as a rigid circular cylinder of infinite extent. The formulas extend a previously developed model [1] to give a method for making a full propeller noise calculation by including the effect of constant axial flow, chordwise and radial loading distributions, and blade sweep. The sound pressure produced by the interaction of the wake of the upstream rotor with the downstream rotor of a counter-rotating advanced open rotor was considered. It was shown that scattering from the hub had a significant effect on the radiated sound pressure level, and therefore the effect should be included in advanced open-rotor noise prediction methods.

Appendix A: Stationary Phase Evaluation of Far-Field Equation

The integral defined by Eq. (10) can be evaluated by the method of stationary phase by first replacing the Hankel function $H^{(2)}_{-\nu}(\beta r_o)$ by its asymptotic form:

$$H^{(2)}_{-\nu}(\beta r_o) \sim \sqrt{\frac{2}{\pi\beta r_o}}\exp\left\{-i\beta r_o - i\nu\frac{\pi}{2} + i\frac{\pi}{4}\right\}; \quad \beta r_o \to \infty \tag{A1}$$

Defining a new integration variable $\gamma = \bar{\alpha}/\kappa$ (noting that $\kappa = \omega_{mk}/c_0$) and introducing the emission angle θ, which is defined by $\cos\theta = M_x - x_o/|\mathbf{x}_o|$ and $\sin\theta = r_o/|\mathbf{x}_o|$, the integral can be expressed in the form,

$$I_{\bar{\alpha}} \sim \int_{-\infty}^{\infty} g(\gamma)\exp\{i\kappa|\mathbf{x}_o|h(\gamma)\}\,\mathrm{d}\gamma \tag{A2}$$

where

$$h(\gamma) = \left[-\gamma(M_x - \cos\theta) - \frac{\beta}{\kappa}\sin\theta\right]$$

$$g(\gamma) = \kappa\left[\gamma\kappa\sin\alpha - \cos\alpha\frac{\nu}{r}\right]\sqrt{\frac{2}{\pi\beta r_o}}\exp\left\{i\kappa\gamma x - i\nu\frac{\pi}{2} + i\frac{\pi}{4}\right\}\Theta$$

When $\kappa|\mathbf{x}_o| \to \pm\infty$, the integral can be evaluated using the method of stationary phase (see [4]) as

$$I_{\bar{\alpha}} \sim g(\gamma_s)\sqrt{\frac{2\pi}{\kappa|\mathbf{x}_o|h''(\gamma_s)}}\exp\left\{i\kappa|\mathbf{x}_o|h(\gamma_s) + i\frac{\pi}{4}\right\} \tag{A3}$$

where γ_s is the point where $h'(\gamma) = 0$ and is equal to $-\cos\theta/(1 - M_x\cos\theta)$. The integral is thus evaluated as

$$I_{\bar{\alpha}} \sim \frac{-2i}{|\mathbf{x}_o|(1-M_x\cos\theta)}\left[\frac{\omega_{mk}/c_0}{(1-M_x\cos\theta)}\cos\theta\sin\alpha+\cos\alpha\frac{\nu}{r}\right] \times \exp\left\{-i\frac{\omega_{mk}}{c_0}|\mathbf{x}_o| - i\frac{\omega_{mk}/c_0}{1-M_x\cos\theta}x\cos\theta + i\nu\frac{\pi}{2}\right\}\bar{\Theta} \tag{A4}$$

where $\bar{\Theta}$ is defined in Eq. (18).

Appendix B: Green's Function Derivation

We consider a rigid cylinder of radius b with the axis collinear with the x axis. The fluid flows uniformly in the positive x direction, with a velocity of U_x and a Mach number of $M_x = U_x/c_0$. We place a single frequency source, with $\exp\{i\omega t\}$ time dependence, at the point $\mathbf{x} = \{x, r, \phi\}$: the resulting pressure field at $\mathbf{x}_o = \{x_o, r_o, \phi_o\}$ is the Green's function defined by

$$\left[\left(\frac{1}{c_0}\frac{\partial}{\partial t} + M_x\frac{\partial}{\partial x_o}\right)^2 - \frac{1}{r_o}\frac{\partial}{\partial r_o}\left(r_o\frac{\partial}{\partial r_o}\right) - \frac{1}{r_o^2}\frac{\partial^2}{\partial\phi_o^2} - \frac{\partial^2}{\partial x_o^2}\right]\tilde{G}(\mathbf{x}_o|\mathbf{x})\exp\{i\omega t\} = \delta(\mathbf{x}_o-\mathbf{x})\exp\{i\omega t\} \tag{B1}$$

with a rigid flow boundary condition on the cylinder surface and a radiation condition as $r_o \to \infty$. A Fourier transform is defined by

$$\tilde{G}_n(\bar{\alpha}, r_o) = \int_{-\infty}^{\infty}\int_{-\pi}^{\pi}\tilde{G}(\mathbf{x}_o|\mathbf{x})\exp\{i\bar{\alpha}x_o + in\phi_o\}\,\mathrm{d}\phi_o\,\mathrm{d}x_o \tag{B2}$$

$$\tilde{G}(\mathbf{x}_o|\mathbf{x}) = \frac{1}{(2\pi)^2}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{G}_n(\bar{\alpha}, r_o)\exp\{-i\bar{\alpha}x_o - in\phi_o\}\,\mathrm{d}\bar{\alpha} \tag{B3}$$

Applying these expressions to Eq. (B1) gives

$$\left[\frac{1}{r_o}\frac{\partial}{\partial r_o}\left(r_o\frac{\partial}{\partial r_o}\right) + \beta^2 - \frac{n^2}{r_o^2}\right]\tilde{G}_n = -\frac{\delta(r_o-r)}{r}\exp\{i\bar{\alpha}x + in\phi\} \tag{B4}$$

It can be shown (see Morse and Feshbach [5]) that for the general second-order ordinary differential equation:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}z_o^2} + a_1(z_o)\frac{\mathrm{d}}{\mathrm{d}z_o} + a_0(z_o)\right]f(z_o) = g(z_o)\delta(z_o - z) \tag{B5}$$

The solution can be written as

$$f(z_o) = \frac{-g(z)f^+(z_>)f^-(z_<)}{W[f^+(z)f^-(z)]} \tag{B6}$$

where $f^{\pm}$ are solutions of the homogeneous equation, f^+ obeys the boundary condition for $z_o > z$, f^- obeys the boundary condition for $z_o < z$, and W denotes the Wronskian,

$$W[f^+(z_o), f^-(z_o)] = f^+(z_o)\frac{\mathrm{d}f^-}{\mathrm{d}z_o} - f^-(z_o)\frac{\mathrm{d}f^+}{\mathrm{d}z_o} \tag{B7}$$

and we define

$$z_> = \left\{\begin{matrix} z_o & \text{if } z_o > z \\ z & \text{if } z_o < z\end{matrix}\right\}; \qquad z_< = \left\{\begin{matrix} z & \text{if } z_o > z \\ z_o & \text{if } z_o < z\end{matrix}\right\} \tag{B8}$$

The Green's function is thus given by

$$\tilde{G}_n(\bar{\alpha}, r_o) = \frac{\exp\{i\bar{\alpha}x + in\phi\}}{r}\frac{\tilde{G}_n^+(\bar{\alpha}, r_>)\tilde{G}_n^-(\bar{\alpha}, r_<)}{W[\tilde{G}_n^+(\bar{\alpha}, r)\tilde{G}_n^-(\bar{\alpha}, r)]} \tag{B9}$$

where $\tilde{G}_n^{\pm}$ are solutions of the homogeneous equation, with $\tilde{G}_n^+$ obeying the radiation condition as $r_o \to \infty$, and $\tilde{G}_n^-$ obeying the boundary condition at $r_o = b$.

The functions $\tilde{G}_n^{\pm}$ are defined by

$$\left[\frac{1}{r_o}\frac{\partial}{\partial r_o}\left(r_o\frac{\partial}{\partial r_o}\right) + \beta^2 - \frac{n^2}{r_o^2}\right]\tilde{G}_n^{\pm} = 0 \tag{B10}$$

which is simply a form of Bessel's equation. For outgoing waves, we have

$$\tilde{G}_n^+ = H_n^{(2)}(\beta r_o) \tag{B11}$$

and applying the boundary condition on the cylinder surface gives

$$\tilde{G}_n^- = J_n(\beta r_o) - A_n(\bar{\alpha}, \omega)H_n^{(2)}(\beta r_o); \quad A_n(\bar{\alpha}, \omega) = \frac{J_n'(\beta b)}{H_n^{(2)\prime}(\beta b)} \tag{B12}$$

The Wronskian evaluates as $2i/\pi r$. Substituting the expressions for the Wronskian and $\tilde{G}_n^{\pm}$ into Eq. (B9) and making use of Eq. (B3) gives

$$\tilde{G}(\mathbf{x}_o|\mathbf{x}) = \frac{-i}{8\pi}\sum_{n=-\infty}^{\infty}\exp\{-in(\phi_o-\phi)\}\int_{-\infty}^{\infty}[J_n(\beta r_<) - A_n(\bar{\alpha}, \omega)H_n^{(2)}(\beta r_<)]H_n^{(2)}(\beta r_>)\exp\{-i\bar{\alpha}(x_o-x)\}\,\mathrm{d}\bar{\alpha} \tag{B13}$$

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A. Sinha
Associate Editor